Indian Statistical Institute, Bangalore
M. Math. Second Year
Second Semester - Graph Theory and Combinatorics
Final Exam Duration : 3 hours Max Marks 100 Date : April 24, 2017

Remark: Each question carries 20 marks. Answer any five questions.

1. Let $n \geq 3$ and let X be a family of n mutually orthogonal partitions of a set of size n^2.
   (a) Show that each member of X consists of n cells of size n each.
   (b) Show that X extends uniquely to a family of $n+1$ mutually orthogonal partitions of its ground set.
2. (a) Prove that the dual of any oval in a projective plane of odd order is an oval in the dual plane.
   (b) Count the total number of hyper ovals in a projective plane of order 4.
3. (a) Classify up to isomorphism all connected graphs (finite or infinite) with maximum degree ≤ 2.
   (b) Use part (a) to prove the Schroeder - Bernstein theorem.
4. Let G be a minimal imperfect graph of order n.
   (a) Show that G has exactly n cliques of size $w(G)$ and n cocliques of size $\alpha(G)$.
   (b) Show that each vertex of G is in $\alpha(G)$ cocliques of size $\alpha(G)$ and in $w(G)$ cliques of size $w(G)$.
5. (a) Show that any square 2 - design of order n on $4n-1$ points is either a Hadamard design or its complement.
   (b) Let $d = 4n-1$. Show that there is a Hadamard matrix of order $4n$ iff there is a regular simplex Δ inscribed in the hypercube $C = [0,1]^d$ in $\mathbb{R}^d$, such that each vertex of Δ is a vertex of C. (A regular simplex in $\mathbb{R}^d$ is the convex hull of $d+1$ equidistant points.)
6. (a) Classify all extended Dynkin diagrams, i.e., connected graphs with Perron eigenvalue $= 2$.
   (b) Hence classify all Dynkin diagrams, i.e., connected graphs with Perron eigenvalue < 2.